Problem Set 4

Due: Wednesday, October 12 2005.

Problem 1. In class we proved a running time bound of $O(m^{3/2})$ for finding a maximum flow in a unit-capacity graph with m edges using blocking flows. We will prove some related bounds.

(a) Consider an arbitrary unit-capacity graph, and let d be the distance between the source and sink. Prove that the value of the maximum flow is at most n^2/d^2. Conclude that $O(n^{2/3})$ blocking flows in time $O(mn^{2/3})$ suffice to find a maximum flow. **Hint:** argue that there are two adjacent layers in the admissible graph with few vertices in each. How much flow can cross between the two layers?)

(b) Consider a graph in which all edges are unit capacity, *except* for the edges incident on p vertices which have arbitrary integer capacities. Extend the unit-capacity flow bounds to give good bounds in terms of m, n, and p on the number of blocking flows needed to find a maximum flow.

Problem 2. You are TAing a class with s students and r recitations. The students tell you which recitations they can fit into their schedules. However, each recitation has room for only k students. (Conversely, each student only has to attend one recitation.) Your goal is to assign the most students to recitations within these constraints.

(a) Suppose first that $k = 1$ (personal tutors!). How can you easily solve this problem in $O(m\sqrt{n})$ time?

(b) Now suppose k is arbitrary. How can you solve this problem with maximum flow?

(c) Argue that an $O(m\sqrt{n})$ time bound can be achieved for your network from part (b), as in unit networks.

Problem 3. Consider a matrix of numeric data where each entry is fractional, but each row and column sum is an integer. Prove that you can "round off" this matrix, rounding each entry to the next integer above or below, *without* changing the row or column sums. **Hint:** the matrix can be thought of as describing (most of) a flow in a bipartite graph. Recall that integer flow problems have integer solutions.