Problem Set 4

Due: Wednesday, October 19 2005.

NONCOLLABORATIVE Problem 1. Give efficient algorithms for the following variants of min-cost flow.

(a) Find the minimum-cost flow subject to the constraint that the flow value is at least 90% of the maximum flow value.

(b) Find the flow that minimizes the cost of the flow plus K times the difference between the flow value and the maximum flow value, for a given weight K.

Problem 2. Cost-scaling algorithm for minimum-cost flow.

(a) Suppose that you have an optimal solution to some integral minimum-cost circulation problem, along with the feasible prices that produce only nonnegative reduced cost arcs. Now suppose that you then change one edge cost by one unit. Show how you can re-optimize the solution using a max-flow algorithm. **Hint:** given the reduced costs, what can go wrong from the change, and what kind of edges can be involved in the repair?

(b) Deduce a cost-scaling algorithm for minimum-cost flow that makes $O(m \log C)$ calls to your solution to part (a).

OPTIONAL (c) Design a cost-scaling algorithm for minimum-cost flow that makes $O(n \log C)$ calls to your solution to part (a).

Note: The resulting time bounds are worse than the capacity-scaling algorithm we saw in lecture, but a more careful algorithm along these lines attains a better time bound of $O(mn \log \log U \log_n C)$.

Problem 3. You are given a polytope defined by a collection of inequalities $Ax \leq b$, and want to determine the largest ball that will fit inside the polytope. Show how this can be done by writing a linear program to identify the center of that ball.

Problem 4. You have been chosen to assign the new graduate students to office spaces, and want to benefit from the experience. Office i has n_i available desks, and graduate student g is willing to pay you p_{ig} dollars for the privilege of being placed in office i.

(a) Show how you can use a min-cost flow algorithm to work out the best (for you) assignment of new students to offices.

(b) Argue that you can achieve this best result without forcing grad students to share a desk or split their time between multiple rooms.

(c) Write a concise linear program to solve this problem (assuming it returns an integral answer).

Problem 5. You work for the Short-Term Capital Management company and start the day with D dollars. Your goal is to convert them to Yen through a series of currency trades involving assorted currencies, so as to maximize the amount of Yen you end up with. You are given a list of pending orders: client i is willing to convert up to u_i units of currency a_i into currency b_i at a rate of r_i (that is, he will give you r_i units of currency b_i for each unit of currency a_i). Assume that going around any directed cycle of trades, $\prod r_i < 1$—that is, there is no opportunity to make a profit by arbitrage.

(a) Formulate a linear program for maximizing the amount of Yen you have at the end of trading.

(b) Show that it is possible to carry out trades to achieve the objective of the linear program, without ever borrowing currency. (**Hint:** there is a sense in which your solution can be made acyclic.)

(c) Show that there is a sequence of trades that will let you end the day with the optimum amount of Yen and no other currency except dollars.