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Problem Set 1

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Problem 1.2.

Part (a): Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the eigenvalues of A and B respectively. Since $A - B$ is positive semi-definite, we have $x^*Ax \geq x^*Bx$ for all x.

Begin by generalizing the Courant-Fischer theorem. Let $A = U\Lambda U^*$ where U is an orthonormal basis, and let $U^*x = z$ (and thus $x = Uz$), also label A's eigenvalues as $\lambda_1 \leq \cdots \leq \lambda_n$. For an arbitrary choice of vectors $w_1, \ldots, w_{k-1}$ we have:

$$
\begin{aligned}
\min_{\substack{x^*x=1 \\ x\perp w_1,\ldots,w_{k-1}}} x^*Ax &= \min_{\substack{x^*x=1 \\ Uz\perp w_1,\ldots,w_{k-1}}} \sum \lambda_i z_i^2 \\
&= \min_{\substack{z^*z=1 \\ z\perp U^*w_1,\ldots,U^*w_{k-1}}} \sum \lambda_i z_i^2 \\
&\leq \min_{\substack{z^*z=1 \\ z_{k+1}=\cdots=z_n=0 \\ z\perp U^*w_1,\ldots,U^*w_{k-1}}} \sum \lambda_i z_i^2 \qquad (\dagger) \\
&= \min_{\substack{z_1^2+\cdots+z_k^2=1 \\ z_{k+1}=\cdots=z_n=0 \\ z\perp U^*w_1,\ldots,U^*w_{k-1}}} \sum \lambda_i z_i^2 \\
&\leq \lambda_k
\end{aligned}
$$

Note that in order to justify inequality ($\dagger$) we need to make sure that the minimization is not over the empty set. Since $z \perp U^*w_1, \ldots, U^*w_{k-1}$ is an underspecified linear system with at most $k-1$ independent constraints, we can add up to $n-k+1$ consistent constraints (in this case $z_{k+1} = \cdots = z_n = 0$) while ensuring that the system still has a solution. Furthermore, the use of "min" (instead of "inf") is justified since $x^*x = 1$ is compact. Combining this with the Ritz-Rayleigh equality:

$$\lambda_k = \min_{\substack{x^*x=1 \\ x\perp u_1,\ldots,u_{k-1}}} x^*Ax$$

Gives the desired equality:

$$\lambda_k = \max_{w_1,\ldots,w_{k-1}} \min_{\substack{x^*x=1 \\ x_1\perp w_1,\ldots,w_{k-1}}} x^*Ax$$

We now use the latter to show $\lambda_k \geq \mu_k$ for all k:

$$
\begin{aligned}
\lambda_k &= \max_{w_1,\ldots,w_{k-1}} \min_{\substack{x^*x=1 \\ x_1 \perp w_1,\ldots,w_{k-1}}} x^*Ax \\
&\geq \max_{w_1,\ldots,w_{k-1}} \min_{\substack{x^*x=1 \\ x_1 \perp w_1,\ldots,w_{k-1}}} x^*Bx \\
&= \mu_k
\end{aligned}
$$

Part (b): Using $f^*Lf = \sum_{x\sim y} \big(f(x) - f(y)\big)^2$:

$$
\begin{aligned}
f^*L_{P_n}f &= \sum_{i=1}^{n-1} \big(f(x_i) - f(x_{i+1})\big)^2 \\
&\geq \frac{1}{n-1} \cdot \big(f(n) - f(1)\big)^2 \qquad \text{using } (x_1 - x_n)^2 \leq (n-1)\sum (x_i - x_{i+1})^2 \\
&= \frac{f^*L_{1,n}f}{n-1}
\end{aligned}
$$

Hence $(n-1) \cdot f^*L_{P_n}f - f^*L_{1,n}f \geq 0$ and $(n-1)L_{P_n} \succeq L_{1,n}$.

Short discussion: Applying the Laplacian of a graph G to a function f essentially disperses the (appropriately defined) volume of f across G's vertices, using local corrections only. Applying the negative Laplacian of (a possibly different) graph H counteracts this dispersion using local corrections w.r.t. H. Note that f^*Lf is a measure of how much functional volume has been moved in one step. To say that $f^*L_Gf \succeq f^*L_Hf$ is to say that G is faster at dispersing f than H is at counteracting.

Part (c): For simplicity if notation we let L_{j-i} denote the Laplacian of the path graph between i and j on n vertices, whose edges are $(i, i+1), \ldots, (j-1, j)$. Then:

$$
\begin{aligned}
L_{K_n} &= \sum_{j>i} L_{j,i} \\
&\preceq \sum_{i<j} (j-i)L_{j-i} && \text{using Part (b)} \\
&\preceq (n-1)\sum_{i<j} L_{j-i} \\
&\preceq (n-1)\sum_{i<j} L_{n-1} && \text{using } L_H \preceq L_G \text{ when } H \subseteq G \\
&= O(n^3)L_{P_n}
\end{aligned}
$$

The eigenvalue bound now follows from Part (a) and $\lambda_2(K_n) = n$.

Part (d): Let $[n]$ be the vertex set of the tree under consideration T_n. Set L_{j-i} to denote the shortest path between i and j along the tree edges. Such a path is no longer than $2\log n$.

$$\begin{aligned}
L_{K_n} &= \sum_{j>i} L_{j,i} \\
&\preceq 2\log n \sum_{j>i} L_{j-i} \\
&\preceq 2\log n \sum_{e\in E(T_n)} O(n^2) L_e && \text{since each } T_n\text{-edge is used at most } \binom{n}{2} \text{ times} \\
&= O(n^2 \log n) L_{T_n}
\end{aligned}$$

The eigenvalue bound now follows from Part (a) and $\lambda_2(K_n) = n$.

Problem 1.3.

We begin with a slightly different proof of the inequality $\lambda_2(G) \leq O\big(\Phi(G)\big)$, which illuminates how $\lambda_2(G)$ can depart from $\Phi(G)$. Let $L_{G(S)}$ denote the Laplacian of the induced subgraph of G on the vertex subset $S \subseteq V$. Let $L_{K(S)}$ denote the Laplacian of the complete graph on the vertex subset $S \subseteq V$. Also if $H \subseteq V \times V$ is an edge set, then let L_H denote its Laplacian. Set $T = V \backslash S$. For any cut S we have:

$$\begin{aligned}
L_G &= L_{G(S)} + L_{\partial S} + L_{G(T)} \\
&\preceq L_{K(S)} + L_{\partial S} + L_{K(T)} && \text{using } L_H \preceq L_G \text{ when } H \subseteq G \qquad (\dagger) \\
&= L_{K(S)} \oplus L_{K(T)} + L_{\partial S} && \text{since } S \cap T = \emptyset \\
&= L_H && \text{here } H = K(S) \cup \partial S \cup K(T)
\end{aligned}$$

Using Part (a) of Problem 1.2 we know that $\lambda_2(L_G) \leq \lambda_2(L_H)$. Define

$$g(x) = \begin{cases} \frac{|T|}{|T|-|S|}, & \text{if } x \in S \\ \frac{|S|}{|T|-|S|}, & \text{if } x \in T \end{cases}$$

And note that $g \perp \mathbf{1}$. Then:

$$\begin{aligned}
\lambda_2(L_H) &= \min_{f \perp \mathbf{1}} \frac{\sum_{\{x,y\} \in E(H)} \big(f(x) - f(y)\big)^2}{\sum_{x \in V(H)} f(x)^2} \\
&\le \frac{\sum_{\{x,y\} \in E(H)} \big(g(x) - g(y)\big)^2}{\sum_{x \in V(H)} g(x)^2} \\
&= \frac{O(n) \cdot |\partial S|}{|S| \cdot |T|} \\
&= O\left(\frac{|\partial S|}{\min\big(|S|, |T|\big)}\right) \\
&= O\big(\Phi(G)\big)
\end{aligned} \tag{$\ddagger$}$$

Note what we have done: We fixed a sparse cut in the graph, then we "densified" (†) both sides of the cut and than we approximated the second eigenvector (‡) of the denser graph. For the path graph, the densification step considerably departs from P_n's eigenvalue hence the gap between $\lambda(P_n)$ and $\Phi(P_n)$. For the tree graph, it would seem that the densification step changes the eigenvalue, but in fact it does not w.r.t. the approximate eigenfunction used in (‡), hence the tight inequality.

The inequality $O(\Phi(G)^2) \le \lambda(G)$ can be rephrased in a similar fashion. Here we fix an eigenfunction f and then break up long edges into short ones w.r.t. f. Then we substitute f with a slightly less-conducted function g whose eigenvalue can be related to the sparsest cut (in the fixed order). The subtlety in this latter step is that g is analyzed as if all (ordered) cuts on g are as small as the smallest one!

Note that the sinusoidal shape of P_n's eigenvectors induces about $n/2$ long edges, each of which is cut into 2 short ones. This has little impact on conductance. More importantly though, all cuts along the eigenvector cut the same number of edges, which is 2. Thus the Cheeger inequality is tight up to a constant.

For the tree, we can see that there are about $n/\log n$ long edges, each cut into (on average) $\log n$ short ones. This is does not have significant impact on the eigenvalue. However, the majority of the cuts along the eigenvector cut $O(n \log n)$ short edges, while the smallest cut cuts only 1 edge. Therefore, the analysis grossly underestimates the Cheeger constant, while still managing to find it algorithmically.

It is quite intuitive now that $P_n \times T_n$ can give a good intermediate case. Why? We have $\lambda(P_n \times T_n) = O(1/n)$ and $\Phi(P_n \times T_n) = O(1/n)$. The first inequality is tight for the same reason it is tight for the tree. (Note that there are at least two natural optimal cuts, for both of which this argument works. One natural cut is to cut all trees at the center. The other is to cut all paths at the center.) Let's consider what the grid eigenvector does. Restricting the eigenvector to any path or tree produces the correct ordering. However, it is not hard to see that all trees are displaced in the global order) with respect to each other such that every tree-wise cut is badly dense. Likewise, all paths are displaced from each other by the tree eigenvector, so that every cut corresponds to a zig-zag cut through the paths.