6.895 Randomness and Computation March 20, 2006

Homework 3

Lecturer: Ronitt Rubinfeld *Due Date: April 10, 2006*

1. Show that the cover time of the random walk on the n-node line is $O(n^2)$.

2. You are given a 2-SAT formula $\phi(x_1, \ldots, x_n)$. Consider the following algorithm for finding a satisfying assignment:

   - Start with an arbitrary assignment. If it's satisfying, output it and halt.
   - Do s times:
     - Pick an arbitrary unsatisfied clause
     - Pick one of the two literals in it uniformly at random
     - Complement the setting of the chosen literal
     - If the new assignment satisfies ϕ, output the assignment and halt.

   Show that if you pick s to be $O(n^2)$, you will output a satisfying assignment with probability at least 3/4.

3. Let $\mathcal{G}_{d,n}$ be the class of d-regular undirected graphs on n nodes (that is, each node has degree exactly d). Say that a graph $G \in \mathcal{G}_{d,n}$ is labeled if, at each vertex in the graph, each of the d edges incident on that vertex has a unique integer label in $\{1, \ldots, d\}$. There is no requirement that an edge receive the same label at both end-points.

   Any sequence of symbols $\sigma =< \sigma_1, \sigma_2, \ldots, \sigma_m >$ for $\sigma_i \in \{1, \ldots, d\}$ together with a start vertex v describes a walk through the labelled graph in the following natural fashion – the walk begins at v, and at each step i follows the edge labelled σ_i. Given a start node, a sequence σ traverses G if the walk visits each vertex of G. A sequence σ is a *universal traversal sequence* for $\mathcal{G}_{d,n}$ if it traverses every labeled graph in $\mathcal{G}_{d,n}$ from every possible start node.

   (a) Show that the number of labelled graphs from $\mathcal{G}_{d,n}$ is $(nd)^{O(nd)}$

   (b) Show that there is a sequence σ which is a univeral traversal sequence for $\mathcal{G}_{d,n}$ such that $|\sigma|$ (i.e., the length of σ) is $O(n^3 \text{poly}(d) \log n)$.

4. Given graph G (regular, undirected and not bipartite). Let A be a random walker that is starting at the uniform (stationary) distribution on nodes. Let B be a random walker that starts at an arbitrary node u. Let M_G^u be the expected time for A to meet up with B. Give the best general upper and lower bounds that you can on M_G^u.

5. Consider the following card shuffling procedure:

   - pick a random card in the deck
   - move it to the top

   Consider the Markov chain M in which each node represents a card shuffle, and edges are between nodes that can reach each other via one step of the above procedure.

- Why is M connected?
- Find a coordinator strategy that minimizes the coupling time (it should at least be bounded by a polynomial).