6.895 Randomness and Computation April 10, 2006

Homework 4

Lecturer: Ronitt Rubinfeld *Due Date: April 24, 2006*

1. Let P be the transition matrix of a random walk on an undirected graph with eigenvectors $v_1, \ldots, v_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.
   (a) For each of the following, what are the eigenvectors and eigenvalues (in terms of those of P)? Justify your answer.
   - αP
   - $P + I$
   - P^k
   - $((P + I)/2)^k$

   (b) Show that since P is stochastic, all eigenvalues $|\lambda_i| \leq 1$.
   (c) if $w = \sum_i \alpha \cdot v^{(i)}$, where the $v^{(i)}$s are eigenvectors of P that form an orthonormal basis, show that $||w|| = \sqrt{\sum \alpha_i^2}$.

2. Let $G(V, E)$ be a graph with n vertices such that for some constant $\alpha > 0$, and every set $S \subseteq V$ with $n/2$ vertices,
$$|\{w \in V | \exists v \in S, (v, w) \in E\}| \geq \frac{n}{2} + \alpha n.$$
For any positive integer k, let $W_1, \ldots, W_k$ be subsets of V of size at least $(1 - \alpha)n$ each. Show that there exists a path $(v_1, \ldots, v_k)$ in G such that for $1 \leq i \leq k$, $v_i \in W_i$.

3. A graph has (K, A)-vertex expansion if $\forall S \subset V, |S| \leq K, |\lambda(S)| \geq A|S|$ where $\lambda(S) = |\{u | \exists v \in S s.t. (u, v) \in E\}|$. For a probability distribution π over $[n]$, the *collision probability* is
$$||\pi||^2 = \sum_x \pi_x^2$$
Show that the following is true:
   - $||\pi||^2 \geq \frac{1}{|S(\pi)|}$ where $S(\pi) = \{x | \pi_x > 0\}$.
   - $||\pi||^2 = ||\pi - u||^2 + \frac{1}{n}$ where u is the uniform distribution.
   - If there is a constant $\lambda < 1$ such that the transition matrix of G is such that $\lambda_2 \leq \lambda$ then for any $\alpha < 1$, G has vertex expansion $(\alpha n, \frac{1}{(1-\alpha)\lambda^2 + \alpha})$.

4. (optional) Let $G = (V, E)$ be a d-regular, α-expander, i.e., $\forall S \subset V, |S| \leq |V|/2, |\lambda(S) \backslash S| \geq \alpha |S|$ (λ is defined as above). Let $n = |V|$. Suppose a set of pairs $S = \{(a_1, b_1), \ldots, (a_q, b_q)\}$ are such that a_i and b_i are chosen uniformly and independently from V for all i. Show that for all a, b, there is a path connecting a, b of length $O(\log n)$. Show that there exists a way of connecting each a_i, b_i pair via a path of length $O(\log n)$, such that no edge is used more than $O(\log n)$ times in total over all paths.

   Hint: for each (a_i, b_i) pair, choose a random x_i. Show how to pick a "random" path from a_i to x_i and from b_i to x_i. Show that your method is such that no edge is used more than $O(\log n)$ times in total.