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MIT 6.859 – Randomness and Computation, with Ronitt Rubinfeld
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Problem Set 2 – Solutions

Problem 1

(a) Assume f, h and g are linear consistent, then there is some linear $\phi(\cdot)$ such that:

$$\begin{aligned} f(x) &= \phi(x) + a_f \\ g(y) &= \phi(y) + a_g \\ h(x+y) &= \phi(x+y) + a_h \end{aligned}$$

where $a_f + a_g = a_h$, therefore:

$$\begin{aligned} f(x) + g(y) &= \phi(x) + a_f + \phi(y) + a_g \\ &= \phi(x+y) + a_h \\ &= h(x+y) \end{aligned}$$

In the other direction, define $\phi(x) = h(x) - h(0)$. Check that $\phi(\cdot)$ is linear:

$$\begin{aligned} \phi(x) + \phi(y) &= h(y) - h(0) + h(y) - h(0) \\ &= f(x) + g(0) - h(0) + f(0) + g(y) - h(0) \\ &= \big(f(x) + g(y)\big) + \big(f(0) + g(0)\big) - 2h(0) \\ &= h(x+y) + h(0) - 2h(0) \\ &= h(x+y) - h(0) \\ &= \phi(x+y) \end{aligned}$$

For f, define $a_f = h(0) - g(0)$, then:

$$f(x) = h(x) - g(0) = h(x) - h(0) + a_f = \phi(x) + a_f$$

For g, define $a_g = h(0) - f(0)$, then:

$$g(x) = h(x) - f(0) = h(x) - h(0) + a_g = \phi(x) + a_g$$

For h, define $a_h = h(0)$, then:

$$h(x) = h(x) - h(0) + h(0) = h(x) - h(0) + a_h = \phi(x) + a_h$$

Finally, verify that:

$$\begin{aligned} a_f + a_g &= h(0) - g(0) + h(0) - f(0) \\ &= 2h(0) - \big(f(0) + g(0)\big) \\ &= h(0) \\ &= a_h \end{aligned}$$

(b) We know that $d(f, \chi_S) = (1 - \hat{f}(S))/2$, we also know that $d(f, -\chi_S) = 1 - d(f, \chi_S)$, therefore $d(f, -\chi_S) = (1 + \hat{f}(S))/2$. Hence, we can compute:

$$\begin{aligned} \min_S d(f, \pm\chi_S) &= \min\left\{\min_S d(f, \chi_S), \min_S d(f, -\chi_S)\right\} \\ &= \min\left\{\frac{1}{2}\left(1 - \max_S \hat{f}(S)\right), \frac{1}{2}\left(1 + \min_S \hat{f}(S)\right)\right\} \\ &= \frac{1}{2}\left(1 - \max_S |\hat{f}(S)|\right) \end{aligned}$$

And therefore (we will need this later):

$$\min_S d(f, \pm\chi_S) \leq \delta \Leftrightarrow \max_S |\hat{f}(S)| \geq 1 - 2\delta$$

(c) Begin by observing that:

$$\mathbf{I}\big[f(x)g(y) \neq h(xy)\big] = \frac{1 - f(x)g(y)h(xy)}{2}$$

Furthermore, (following the lecture notes):

$$\begin{aligned}
\mathbf{E}_{x,y}\left[f(x)g(y)h(xy)\right] &= \mathbf{E}_{x,y}\left[\sum_S \hat{f}(S)\chi_S(x)\sum_T \hat{g}(T)\chi_T(y)\sum_U \hat{h}(U)\chi_U(xy)\right] \\
&= \sum_{S,T,U} \hat{f}(S)\hat{g}(T)\hat{h}(U)\mathbf{E}_{x,y}\big[\chi_S(x)\chi_T(y)\chi_U(xy)\big] \\
&= \sum_S \hat{f}(S)\hat{g}(S)\hat{h}(S)
\end{aligned}$$

Notice:

$$\begin{aligned}
\mathbf{Pr}_{x,y}\big[f(x)g(y) \neq h(xy)\big] &= \mathbf{Pr}_{x,y}\Big[\mathbf{I}\big[f(x)g(y) \neq h(xy)\big] = 1\Big] \\
&= \mathbf{E}_{x,y}\Big[\mathbf{I}\big[f(x)g(y) \neq h(xy)\big]\Big] \\
&= \mathbf{E}_{x,y}\left[\frac{1 - f(x)g(y)h(xy)}{2}\right]
\end{aligned}$$

Therefore $\mathbf{Pr}_{x,y}\big[f(x)g(y) \neq h(xy)\big] \leq \delta$ amounts to:

$$\sum_S \hat{f}(S)\hat{g}(S)\hat{h}(S) \geq 1 - 2\delta$$

On the other hand, we derive that:

$$\begin{aligned}
\max_S |\hat{f}(S)|^2 &= \max_S |\hat{f}(S)|^2 \left(\sum_T |\hat{g}(T)|^2\right)\left(\sum_T |\hat{h}(T)|^2\right) && \text{by Parseval} \\
&\geq \max_S |\hat{f}(S)|^2 \left(\sum_T |\hat{g}(T)| \cdot |\hat{h}(T)|\right)^2 && \text{by Cauchy-Schwarz} \\
&= \left(\max_S |\hat{f}(S)| \sum_T |\hat{g}(T)| \cdot |\hat{h}(T)|\right)^2 \\
&\geq \left(\sum_S \big|\hat{f}(S)\hat{g}(S)\hat{h}(S)\big|\right)^2 \\
&\geq \left(\sum_S \hat{f}(S)\hat{g}(S)\hat{h}(S)\right)^2
\end{aligned}$$

Taking the square root gives us:

$$\max_S |\hat{f}(S)| \geq \sum_S \hat{f}(S)\hat{g}(S)\hat{h}(S) \geq 1 - 2\delta$$

Which further implies that:

$$\min_S d(f, \pm\chi_S) \leq \delta$$

The proof for g and h is identical.

Problem 2

Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be monotone, and for $x \in \{\pm 1\}^n$, let $x^+ \in \{\pm 1\}^n$ be x with the i-th entry set to $+1$, and define x^- accordingly. Then we have that:

$$\begin{aligned}\mathbf{I}[f(x^+) \neq f(x^-)] &= \frac{1}{2}\big(f(x^+) - f(x^-)\big) \\ &= \frac{1}{2}\big(f(x^+)\chi_{\{i\}}(x^+) + f(x^-)\chi_{\{0\}}(x^-)\big)\end{aligned}$$

Apply this in the following:

$$\begin{aligned}\mathrm{Inf}_i(f) &= \mathbf{Pr}_x[f(x) \neq f(x \cdot u_i)] \\ &= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \mathbf{I}[f(x^+) \neq f(x^-)] \\ &= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \frac{1}{2}\big(f(x^+)\chi_{\{i\}}(x^+) + f(x^-)\chi_{\{0\}}(x^-)\big) \\ &= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x)\chi_{\{i\}}(x) \\ &= \hat{f}(\{i\})\end{aligned}$$

Problem 3

For any monotone function we have:

$$\begin{aligned}
\inf(f) &= \sum_{i\in[n]} \inf_i(f) \\
&= \sum_{i\in[n]} \hat{f}(\{i\}) \quad \text{since } f \text{ monotone} \\
&= \sum_{i\in[n]} \frac{1}{2^n} \sum_{x\in\{\pm1\}^n} f(x)\cdot x_i \\
&= \frac{1}{2^n} \sum_{x\in\{\pm1\}^n} \sum_{i\in[n]} f(x)\cdot x_i
\end{aligned}$$

Maximizing over all (not just monotone) functions:

$$\begin{aligned}
\max_f \inf(f) &= \max_f \frac{1}{2^n} \sum_{x\in\{\pm1\}^n} \sum_{i\in[n]} f(x)\cdot x_i \\
&= \frac{1}{2^n} \sum_{x\in\{\pm1\}^n} \max_{f(x)} \sum_{i\in[n]} f(x)\cdot x_i
\end{aligned}$$

The last quantity, when n is odd, is clearly maximized only when:

$$f(x) = \text{maj}\ \{x_1, \ldots, x_n\}$$

Problem 4

Let the graph be random d-regular on the left (with each edge connected uniformly at random on the right). Then let X be a random variable equal to the number of subsets of the left vertices that shrink, i.e. whose neighbor sets on the right are equal or smaller in size. For a fixed left set of size k we have that the probability, $p(k)$, that it shrinks is bounded by (using a union bound over all right subsets of size k):

$$p(k) \leq \binom{2n/3}{k} \left(\frac{k}{2n/3}\right)^{kd}$$

Hence, for $\mathbf{E}[X]$ we have:

$$\begin{aligned}
\mathbf{E}[X] &\leq \sum_{k=1}^{n/2} \binom{n}{k} \binom{2n/3}{k} \left(\frac{k}{2n/3}\right)^{kd} \\
&\leq \sum_{k=1}^{n/2} \left(e^2 \frac{3}{2} \left(\frac{3k}{2n}\right)^{d-2}\right)^k \quad \text{using } \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k \\
&< \sum_{k=1}^{\infty} \left(e^2 \frac{3}{2} \left(\frac{3}{4}\right)^{d-2}\right)^k
\end{aligned}$$

When $d > \ln(3e^2)/\ln(4/3) + 2$, we have that $e^2 \frac{3}{2} \left(\frac{3}{4}\right)^{d-2} < 1/2$, and therefore:

$$\mathbf{E}[X] < \sum_{k=1}^{\infty} (1/2)^k < 1$$

Therefore, there exists a graph of left degree $d = \lceil \ln(3e^2)/\ln(4/3) + 3 \rceil$ for which no left subset of size up to $n/2$ shrinks on the right.